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THE POINCARÉ SERIES OF THE POLYNOMIALS  
INVARIANT UNDER  $SU_2$  IN ITS IRREDUCIBLE  
REPRESENTATION OF DEGREE  $\leq 17$

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The Poincaré series of the polynomials invariant under  $SU_2$  in its irreducible representation of degree  $\leq 17$

by

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#### ABSTRACT

Recently T.A. Springer established a formula for the Poincaré series of the invariant polynomials of the compact Lie group  $SU_2$  corresponding to its irreducible representation of given degree  $d+1$ . The formula expresses the Poincaré series as a rational function that is presented as a sum of several other straightforwardly computable rational functions. By use of a computer these functions have been determined for all  $d \leq 16$ .

KEY WORDS & PHRASES: Poincaré series,  $SU_2$ , invariant theory



## INTRODUCTION

Springer's formula (cf. [3]) for the Poincaré series  $P_d(z)$  of the  $SU_2$ -invariant polynomials associated with its  $(d+1)$ -dimensional irreducible representation reads as follows.

$$P_d(z) = \sum_{0 \leq j < \frac{d}{2}} (-1)^j \phi_{d-2j} \left( \frac{(1-z^2)^j z^{j(j+1)}}{(j, z^2)! (d-j, z^2)!} \right),$$

where for  $n \in \mathbb{N}$  the operator  $\phi_n$  transforms a rational function  $f$  in  $z$  to the rational function  $\phi_n f$  determined by

$$(\phi_n f)(z^n) = \frac{1}{n} \sum_{j=1}^n f(e^{2\pi i j/n} z),$$

and where  $(n, z)!$  is the polynomial function given by  $(n, z)! = (1-z)(1-z^2) \dots (1-z^n)$ .

About the Calculation

Clearly, if  $f = \sum_{j=0}^N f_j z^j$  is a polynomial function in  $z$ , then we have

$$(1) \quad (\phi_n f) = \sum_{j=0}^{\lfloor N/n \rfloor} f_{jn} z^j,$$

where  $\lfloor N/n \rfloor$  is the greatest integer  $\leq N/n$ .

Suppose that  $fg$  is the product of two rational functions one of which, say  $g$ , is a function in  $z^n$ , in other words there exists a function  $h$  in  $z$  such that  $h(z^n) = g(z)$ . Then

$$(2) \quad \phi_n(fg) = h\phi_n(f)$$

As the denominator of the  $\phi_{d-2j}$ -argument in a summand of  $P_d(z)$  is a product of factors of the form  $1-z^m$ , we may, after suitable multiplication of denominator and numerator by common factors, assume that the denominator is a product of factors having the form  $1-z^{(d-2j)m}$ . Application of (2) reduces the summand in  $P_d(z)$  to the quotient of the  $\phi_{d-2j}$ -image of a polynomial by

a product of terms of the form  $1-z^m$ .

Using (1), the numerator is readily computed, whereafter we have obtained a summand that is expressed as a rational function with integral coefficients. Addition of these summands (while keeping track of the signs  $(-1)^j$  that appear in the formula) establishes the results.

RESULTS. The  $P_d(z)$  for  $d \leq 16$  are listed in the table below in a so-called representative form (cf. [4]). The interested reader may find the individual summands in the appendix.

TABLE

The Poincaré series of the invariants under  $SU_2$  in its irreducible representation of degree  $d+1 \leq 17$ .

$d = 2$	$\frac{1}{1-z^2}$
$d = 3$	$\frac{1}{1-z^4}$
$d = 4$	$\frac{1}{(1-z^2)(1-z^3)}$
$d = 5$	$\frac{1+z^{18}}{(1-z^4)(1-z^8)(1-z^{12})}$
$d = 6$	$\frac{1+z^{15}}{(1-z^2)(1-z^4)(1-z^6)(1-z^{10})}$
$d = 7$	$\frac{1+2z^8+4z^{12}+4z^{14}+5z^{16}+9z^{18}+6z^{20}+9z^{22}+8z^{24}+9z^{26}+6z^{28}+9z^{30}+5z^{32}+4z^{34}+4z^{36}+2z^{40}+z^{48}}{(1-z^4)(1-z^8)(1-z^{12})(1-z^{12})(1-z^{20})}$
$d = 8$	$\frac{1+z^8+z^9+z^{10}+z^{18}}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)(1-z^7)}$

$$\begin{aligned}
 d = 9 \quad & 1+z^4+5z^8+4z^{10}+17z^{12}+20z^{14}+47z^{16}+61z^{18}+97z^{20}+120z^{22}+ \\
 & 165z^{24}+189z^{26}+223z^{28}+241z^{30}+254z^{32}+254z^{34}+241z^{36}+ \\
 & 223z^{38}+189z^{40}+165z^{42}+120z^{44}+97z^{46}+61z^{48}+47z^{50}+20z^{52}+ \\
 & 17z^{54}+4z^{56}+5z^{58}+z^{62}+z^{66} \\
 & \hline
 & (1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})
 \end{aligned}$$

$$\begin{aligned}
 d = 10 \quad & 1+2z^6+4z^8+4z^9+7z^{10}+8z^{11}+15z^{12}+15z^{13}+20z^{14}+27z^{15}+29z^{16}+ \\
 & 35z^{17}+40z^{18}+44z^{19}+47z^{20}+55z^{21}+52z^{22}+57z^{23}+56z^{24}+57z^{25}+ \\
 & 52z^{26}+55z^{27}+47z^{28}+44z^{29}+40z^{30}+35z^{31}+29z^{32}+27z^{33}+20z^{34}+ \\
 & 15z^{35}+15z^{36}+8z^{37}+7z^{38}+4z^{39}+4z^{40}+2z^{42}+z^{48} \\
 & \hline
 & (1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^8)(1-z^9)(1-z^{10})(1-z^{14})
 \end{aligned}$$

$$\begin{aligned}
 d = 11 \quad & 1+z^4+10z^8+12z^{10}+57z^{12}+95z^{14}+233z^{16}+394z^{18}+745z^{20}+ \\
 & 1173z^{22}+1925z^{24}+2814z^{26}+4139z^{28}+5669z^{30}+7664z^{32}+ \\
 & 9821z^{34}+12390z^{36}+14953z^{38}+17692z^{40}+20204z^{42}+22537z^{44}+ \\
 & 24378z^{46}+25746z^{48}+26413z^{50}+26413z^{52}+25746z^{54}+24378z^{56}+ \\
 & 22537z^{58}+20204z^{60}+17692z^{62}+14953z^{64}+12390z^{66}+9821z^{68}+ \\
 & 7664z^{70}+5669z^{72}+4139z^{74}+2814z^{76}+1925z^{78}+1173z^{80}+745z^{82}+ \\
 & 394z^{84}+233z^{86}+95z^{88}+57z^{90}+12z^{92}+10z^{94}+z^{98}+z^{102} \\
 & \hline
 & (1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18}) \\
 & (1-z^{20})
 \end{aligned}$$

$$\begin{aligned}
 d = 12 \quad & 1+z^3+z^4+3z^6+5z^7+7z^8+11z^9+21z^{10}+25z^{11}+40z^{12}+55z^{13}+75z^{14}+ \\
 & 96z^{15}+130z^{16}+156z^{17}+194z^{18}+228z^{19}+267z^{20}+294z^{21}+330z^{22}+ \\
 & 345z^{23}+364z^{24}+365z^{25}+364z^{26}+345z^{27}+330z^{28}+294z^{29}+ \\
 & 267z^{30}+228z^{31}+194z^{32}+156z^{33}+130z^{34}+96z^{35}+75z^{36}+55z^{37}+ \\
 & 40z^{38}+25z^{39}+21z^{40}+11z^{41}+7z^{42}+5z^{43}+3z^{44}+z^{46}+z^{47}+z^{50} \\
 & \hline
 & (1-z^2)(1-z^4)(1-z^5)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9) \\
 & (1-z^{11})
 \end{aligned}$$

$$\begin{aligned}
d = 13 \quad & 1+z^4+19z^8+32z^{10}+156z^{12}+340z^{14}+916z^{16}+1913z^{18}+4115z^{20}+ \\
& 7741z^{22}+14544z^{24}+25118z^{26}+42489z^{28}+68114z^{30}+106498z^{32}+ \\
& 159697z^{34}+233971z^{36}+331412z^{38}+458878z^{40}+617966z^{42}+ \\
& 814543z^{44}+1047676z^{46}+1321155z^{48}+1629760z^{50}+1972851z^{52}+ \\
& 2341055z^{54}+2728439z^{56}+3120893z^{58}+3509075z^{60}+3876056z^{62}+ \\
& 4210210z^{64}+4496206z^{66}+4723183z^{68}+4879836z^{70}+4960463z^{72}+ \\
& 4960463z^{74}+4879836z^{76}+4723183z^{78}+4496206z^{80}+4210210z^{82}+ \\
& 3876056z^{84}+3509075z^{86}+3120893z^{88}+2728439z^{90}+2341055z^{92}+ \\
& 1972851z^{94}+1629760z^{96}+1321155z^{98}+1047676z^{100}+814543z^{102}+ \\
& 617966z^{104}+458878z^{106}+331412z^{108}+233971z^{110}+159697z^{112}+ \\
& 106498z^{114}+68114z^{116}+42489z^{118}+25118z^{120}+14544z^{122}+ \\
& 7741z^{124}+4115z^{126}+1913z^{128}+916z^{130}+340z^{132}+156z^{134}+ \\
& 32z^{136}+19z^{138}+z^{142}+z^{146} \\
& \hline
& (1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18}) \\
& (1-z^{20})(1-z^{22})(1-z^{24})
\end{aligned}$$

$$\begin{aligned}
d = 14 \quad & 1+z^4+5z^6+3z^7+18z^8+22z^9+55z^{10}+77z^{11}+158z^{12}+227z^{13}+ \\
& 396z^{14}+579z^{15}+908z^{16}+1286z^{17}+1904z^{18}+2609z^{19}+3655z^{20}+ \\
& 4865z^{21}+6502z^{22}+8386z^{23}+10800z^{24}+13480z^{25}+16765z^{26}+ \\
& 20342z^{27}+24480z^{28}+28872z^{29}+33748z^{30}+38709z^{31}+43994z^{32}+ \\
& 49174z^{33}+54374z^{34}+59258z^{35}+63866z^{36}+67854z^{37}+71317z^{38}+ \\
& 73958z^{39}+75790z^{40}+76724z^{41}+76724z^{42}+75790z^{43}+73958z^{44}+ \\
& 71317z^{45}+67854z^{46}+63866z^{47}+59258z^{48}+54374z^{49}+49174z^{50}+ \\
& 43994z^{51}+38709z^{52}+33748z^{53}+28872z^{54}+24480z^{55}+20342z^{56}+ \\
& 16765z^{57}+13480z^{58}+10800z^{59}+8386z^{60}+6502z^{61}+4865z^{62}+ \\
& 3655z^{63}+2609z^{64}+1904z^{65}+1286z^{66}+908z^{67}+579z^{68}+396z^{69}+ \\
& 227z^{70}+158z^{71}+77z^{72}+55z^{73}+22z^{74}+18z^{75}+3z^{76}+5z^{77}+z^{79}+ \\
& z^{83} \\
& \hline
& (1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)(1-z^{10}) \\
& (1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})
\end{aligned}$$



d = 15

$$\begin{aligned}
& 1+2z^4+z^6+32z^8+78z^{10}+378z^{12}+1027z^{14}+3124z^{16}+7672z^{18}+ \\
& 18676z^{20}+40784z^{22}+86197z^{24}+170002z^{26}+323704z^{28}+ \\
& 586665z^{30}+1029080z^{32}+1737391z^{34}+2849501z^{36}+4530000z^{38}+ \\
& 7019920z^{40}+10596388z^{42}+15636688z^{44}+22556344z^{46}+ \\
& 31885923z^{48}+44179876z^{50}+60104681z^{52}+80312694z^{54}+ \\
& 105538484z^{56}+136437227z^{58}+173688963z^{60}+217804048z^{62}+ \\
& 269239746z^{64}+328183158z^{66}+394684584z^{68}+468435861z^{70}+ \\
& 548927416z^{72}+635239454z^{74}+726230157z^{76}+820357306z^{78}+ \\
& 915894836z^{80}+1010799973z^{82}+1102948881z^{84}+1190050285z^{86}+ \\
& 1269884526z^{88}+1340258093z^{90}+1399218002z^{92}+1445037255z^{94}+ \\
& 1476376889z^{96}+1492284743z^{98}+1492284743z^{100}+ \\
& 1476376889z^{102}+1445037255z^{104}+1399218002z^{106}+ \\
& 1340258093z^{108}+1269884526z^{110}+1190050285z^{112}+ \\
& 1102948881z^{114}+1010799973z^{116}+915894836z^{118}+ \\
& 820357306z^{120}+726230157z^{122}+635239454z^{124}+548927416z^{126}+ \\
& 468435861z^{128}+394684584z^{130}+328183158z^{132}+269239746z^{134}+ \\
& 217804048z^{136}+173688963z^{138}+136437227z^{140}+105538484z^{142}+ \\
& 80312694z^{144}+60104681z^{146}+44179876z^{148}+31885923z^{150}+ \\
& 22556344z^{152}+15636688z^{154}+10596388z^{156}+7019920z^{158}+ \\
& 4530000z^{160}+2849501z^{162}+1737391z^{164}+1029080z^{166}+ \\
& 586665z^{168}+323704z^{170}+170002z^{172}+86197z^{174}+40784z^{176}+ \\
& 18676z^{178}+7672z^{180}+3124z^{182}+1027z^{184}+378z^{186}+78z^{188}+ \\
& 32z^{190}+z^{192}+2z^{194}+z^{198}
\end{aligned}$$

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$$\begin{aligned}
& (1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18}) \\
& (1-z^{20})(1-z^{22})(1-z^{24})(1-z^{26})(1-z^{28})
\end{aligned}$$

$$\begin{aligned}
d = 16 \quad & 1+z^3+z^4+2z^5+8z^6+11z^7+30z^8+59z^9+112z^{10}+204z^{11}+389z^{12}+ \\
& 655z^{13}+1146z^{14}+1896z^{15}+3067z^{16}+4823z^{17}+7477z^{18}+11167z^{19}+ \\
& 16506z^{20}+23764z^{21}+33593z^{22}+46576z^{23}+63588z^{24}+85049z^{25}+ \\
& 112273z^{26}+145668z^{27}+186298z^{28}+234773z^{29}+292016z^{30}+ \\
& 357883z^{31}+433493z^{32}+518058z^{33}+611720z^{34}+713572z^{35}+ \\
& 822997z^{36}+937742z^{37}+1057289z^{38}+1178403z^{39}+1299472z^{40}+ \\
& 1417560z^{41}+1530508z^{42}+1634576z^{43}+1728646z^{44}+1808808z^{45}+ \\
& 1873856z^{46}+1921520z^{47}+1951002z^{48}+1960390z^{49}+1951002z^{50}+ \\
& 1921520z^{51}+1873856z^{52}+1808808z^{53}+1728646z^{54}+1634576z^{55}+ \\
& 1530508z^{56}+1417560z^{57}+1299472z^{58}+1178403z^{59}+1057289z^{60}+ \\
& 937742z^{61}+822997z^{62}+713572z^{63}+611720z^{64}+ \\
& 518058z^{65}+433493z^{66}+357883z^{67}+292016z^{68}+234773z^{69}+ \\
& 186298z^{70}+145668z^{71}+112273z^{72}+85049z^{73}+63588z^{74}+ \\
& 46576z^{75}+33593z^{76}+23764z^{77}+16506z^{78}+11167z^{79}+7477z^{80}+ \\
& 4823z^{81}+3067z^{82}+1896z^{83}+1146z^{84}+655z^{85}+389z^{86}+204z^{87}+ \\
& 112z^{88}+59z^{89}+30z^{90}+11z^{91}+8z^{92}+2z^{93}+z^{94}+z^{95}+z^{98} \\
& \hline
& (1-z^2)(1-z^4)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^7)(1-z^8)(1-z^9) \\
& (1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})(1-z^{15})
\end{aligned}$$

#### REMARKS.

- (i) The series  $P_d(z)$  for  $d \leq 10$  and  $d = 12$  appear in [4]. They agree with the contents of the table. Results for  $d \leq 12$  are also available in [1,2], but we have not compared them with ours. The expressions of  $P_d(z)$  for  $13 \leq d \leq 16$  seem to be new.
- (ii) The expression given for  $P_d(z)$  in the appendix has the property that the numerator and denominator are not simultaneously divisible by a polynomial of the form  $(1-z^m)/(1-z^\ell)$  ( $\ell$  divisor of  $m$ ) for  $m$  such that  $1-z^m$  occurs as an explicitly given factor in the denominator. The representative form however is easily obtained from the given one.

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D=2; J=0  
 NUMERATOR:  
 +1Z0  
 DENOMINATOR:  
 (1-Z2)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=2\*\*\*\*\*:  
 NUMERATOR:  
 +1Z0  
 DENOMINATOR:  
 (1-Z2)

D=3; J=0  
 NUMERATOR:  
 +1Z0  
 DENOMINATOR:  
 (1-Z4)(1-Z2)

D=3; J=1  
 NUMERATOR:  
 +1Z2  
 DENOMINATOR:  
 (1-Z2)(1-Z4)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=3\*\*\*\*\*:  
 NUMERATOR:  
 +1Z0  
 DENOMINATOR:  
 (1-Z4)

D=4; J=0  
 NUMERATOR:  
 +1Z0  
 DENOMINATOR:  
 (1-Z1)(1-Z3)(1-Z2)

D=4; J=1  
 NUMERATOR:  
 +1Z1  
 DENOMINATOR:  
 (1-Z1)(1-Z2)(1-Z3)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=4\*\*\*\*\*:  
 NUMERATOR:  
 +1Z0  
 DENOMINATOR:  
 (1-Z2)(1-Z3)

D=5; J=0  
 NUMERATOR:  
 +1Z0+1Z2+4Z4+5Z6+7Z8+4Z10+3Z12  
 DENOMINATOR:  
 (1-Z4)(1-Z6)(1-Z8)(1-Z2)

D=5; J=1  
 NUMERATOR:  
 $+2Z^2+2Z^4+3Z^6+1Z^8+1Z^{10}$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)$

D=5; J=2  
 NUMERATOR:  
 $+1Z^6$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)$

\*\*\*\*\*THE POINCARÉ SERIES FOR D=5\*\*\*\*\*:  
 NUMERATOR:  
 $+1Z^0-1Z^6+1Z^{12}$   
 DENOMINATOR:  
 $(1-Z^4)(1-Z^6)(1-Z^8)$

D=6; J=0  
 NUMERATOR:  
 $+1Z^0+1Z^2+2Z^3+2Z^4+1Z^5+2Z^6$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)(1-Z^{10})$

D=6; J=1  
 NUMERATOR:  
 $+1Z^1+1Z^2+1Z^3+1Z^5$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)(1-Z^{10})$

D=6; J=2  
 NUMERATOR:  
 $+1Z^3$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)(1-Z^{10})$

\*\*\*\*\*THE POINCARÉ SERIES FOR D=6\*\*\*\*\*:  
 NUMERATOR:  
 $+1Z^0-1Z^2-1Z^3+1Z^6+1Z^7-1Z^9$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)(1-Z^{10})$

D=7; J=0  
 NUMERATOR:  
 $+1Z^0+3Z^2+18Z^4+45Z^6+97Z^8+156Z^{10}+236Z^{12}+288Z^{14}+335Z^{16}+329Z^{18}+304Z^{20}+235Z^{22}+174Z^{24}+100Z^{26}+54Z^{28}+20Z^{30}+6Z^{32}$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)(1-Z^{10})(1-Z^{12})(1-Z^{14})$

D=7; J=1  
 NUMERATOR:  
 $+5Z^2+16Z^4+38Z^6+60Z^8+88Z^{10}+93Z^{12}+103Z^{14}+85Z^{16}+69Z^{18}+41Z^{20}+21Z^{22}+5Z^{24}+1Z^{26}$   
 DENOMINATOR:  
 $(1-Z^2)(1-Z^4)(1-Z^6)(1-Z^8)(1-Z^{10})(1-Z^{12})(1-Z^{14})$

D=7; J=2  
 NUMERATOR:

+1Z2+2Z4+9Z6+11Z8+15Z10+15Z12+14Z14+7Z16+6Z18+1Z20  
 DENOMINATOR:  
 (1-Z2)(1-Z4)(1-Z4)(1-Z2)(1-Z6)(1-Z10)

D=7; J=3  
 NUMERATOR:  
 +1Z12  
 DENOMINATOR:  
 (1-Z2)(1-Z4)(1-Z6)(1-Z4)(1-Z6)(1-Z8)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=7\*\*\*\*\*:  
 NUMERATOR:  
 +1Z0-1Z6+2Z8-1Z10+5Z12+2Z14+6Z16+2Z18+5Z20-1Z22+2Z24-1Z26+1Z32  
 DENOMINATOR:  
 (1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)

D=8; J=0  
 NUMERATOR:  
 +1Z0+3Z2+4Z3+7Z4+7Z5+9Z6+8Z7+9Z8+6Z9+5Z10+3Z11+2Z12  
 DENOMINATOR:  
 (1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)

D=8; J=1  
 NUMERATOR:  
 +2Z1+3Z2+5Z3+8Z4+10Z5+10Z6+12Z7+10Z8+8Z9+7Z10+4Z11+1Z12+1Z13  
 DENOMINATOR:  
 (1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)

D=8; J=2  
 NUMERATOR:  
 +1Z2+1Z3+1Z4+1Z6  
 DENOMINATOR:  
 (1-Z1)(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)

D=8; J=3  
 NUMERATOR:  
 +1Z6  
 DENOMINATOR:  
 (1-Z1)(1-Z2)(1-Z3)(1-Z2)(1-Z3)(1-Z4)(1-Z5)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=8\*\*\*\*\*:  
 NUMERATOR:  
 +1Z0-1Z2-1Z3+1Z5+1Z6-1Z11-1Z12+1Z14+1Z15-1Z17  
 DENOMINATOR:  
 (1-Z2)(1-Z2)(1-Z3)(1-Z3)(1-Z4)(1-Z5)(1-Z7)

D=9; J=0  
 NUMERATOR:  
 +1Z0+6Z2+49Z4+160Z6+414Z8+818Z10+1451Z12+2223Z14+3164Z16+4099Z18+5021Z20+5680Z22+6105Z24+6092Z26+5773Z28+5077Z30+4235Z32+3244Z34+2341Z36  
 +1513Z38+893Z40+438Z42+188Z44+53Z46+11Z48  
 DENOMINATOR:  
 (1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)

D=9; J=1  
 NUMERATOR:  
 +11Z2+67Z4+256Z6+670Z8+1452Z10+2629Z12+4268Z14+6160Z16+8247Z18+10101Z20+11636Z22+12411Z24+12531Z26+11742Z28+10395Z30+8488Z32+6490Z34+449

7Z36+2857Z38+1581Z40+766Z42+294Z44+86Z46+13Z48+1750  
 DENOMINATOR:  
 $(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)$

D=9; J=2  
 NUMERATOR:  
 $+3Z2+20Z4+88Z6+214Z8+447Z10+729Z12+1089Z14+1417Z16+1726Z18+1882Z20+1917Z22+1754Z24+1499Z26+1143Z28+812Z30+486Z32+262Z34+1017Z36+327Z38+4Z40$   
 DENOMINATOR:  
 $(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)$

D=9; J=3  
 NUMERATOR:  
 $+1Z4+2Z6+9Z8+11Z10+15Z12+15Z14+14Z16+7Z18+6Z20+1Z22$   
 DENOMINATOR:  
 $(1-Z2)(1-Z4)(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)$

D=9; J=4  
 NUMERATOR:  
 $+1Z20$   
 DENOMINATOR:  
 $(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z6)(1-Z8)(1-Z10)$

\*\*\*\*\*THE POINCARÉ SERIES FOR D=9\*\*\*\*\*:  
 NUMERATOR:  
 $+1Z0+1Z4-1Z6+5Z8+3Z10+18Z12+15Z14+44Z16+43Z18+8Z20+76Z22+122Z24+107Z26+147Z28+119Z30+147Z32+107Z34+122Z36+76Z38+82Z40+43Z42+44Z44+15Z46$   
 $+18Z48+3Z50+5Z52-1Z54+1Z56+1Z60$   
 DENOMINATOR:  
 $(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)$

D=10; J=0  
 NUMERATOR:  
 $+1Z0+1Z1+8Z2+21Z3+50Z4+88Z5+161Z6+244Z7+372Z8+510Z9+678Z10+835Z11+1012Z12+1137Z13+1256Z14+1304Z15+1320Z16+1257Z17+1178Z18+1028Z19+881Z20$   
 $+704Z21+546Z22+390Z23+277Z24+169Z25+104Z26+53Z27+27Z28+9Z29+4Z30$   
 DENOMINATOR:  
 $(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z2)$

D=10; J=1  
 NUMERATOR:  
 $+3Z1+6Z2+15Z3+25Z4+39Z5+55Z6+72Z7+83Z8+98Z9+103Z10+104Z11+98Z12+90Z13+73Z14+60Z15+42Z16+29Z17+17Z18+9Z19+2Z20+1Z21$   
 DENOMINATOR:  
 $(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)(1-Z3)$

D=10; J=2  
 NUMERATOR:  
 $+1Z1+2Z2+9Z3+15Z4+26Z5+35Z6+52Z7+60Z8+74Z9+77Z10+80Z11+72Z12+69Z13+52Z14+42Z15+29Z16+19Z17+8Z18+5Z19+1Z20$   
 DENOMINATOR:  
 $(1-Z1)(1-Z2)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)$

D=10; J=3  
 NUMERATOR:  
 $+1Z3+2Z5+2Z6+3Z7+2Z8+2Z9+1Z10+2Z11+1Z12$   
 DENOMINATOR:  
 $(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)(1-Z7)$

D=10; J=4  
 NUMERATOR:  
 $+1Z10$



DENOMINATOR:  
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z22)(1-Z3)(1-Z4)(1-Z5)(1-Z6)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=10\*\*\*\*\*:

NUMERATOR:  
+170-172-173+376+278+279+4710+5712-1713+1714+3716-4717+4718-3719-1721+1722-6723-4725-2726-2727-3729+1732+1733-1735  
DENOMINATOR:  
(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)

D=11; J=0  
NUMERATOR:  
+170+1342+15324+86176+351628+11108710+29737212+69185214+145026216+277592718+493864220+822988222+1297562724+1944356726+2786068228+3828074  
730+5063386232+6458374734+7964750736+9506816238+11002659240+12353603242+13472597244+14273622746+14702066748+14719445250+14330077752+1355  
8279254+12467535756+11131643258+9647262760+8103500762+6592623764+5183387266+3933427268+2871899770+201314772+1348408774+860020276+518372  
778+293571780+15421782+74416784+32197286+12274288+3912290+1004792+177794+20796  
DENOMINATOR:  
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)

D=11; J=1  
NUMERATOR:  
+222+20174+98276+316478+8046710+17042712+31863714+53525216+83060218+119965720+153532222+211257724+260380726+306277728+345814730+3743387  
32+390158734+390706736+377030738+349333740+311295742+265496744+216902746+168610748+124653750+86697752+56499754+33846756+18424758+8780760  
+3566762+1127264+251266+27268+1270  
DENOMINATOR:  
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)

D=11; J=2  
NUMERATOR:  
+822+9474+56926+210278+6071710+14204712+28952214+52249716+86119218+130619220+185535222+247579724+313406726+376697728+432357730+473515732  
+496598734+497895736+478268738+438986740+385509742+322512744+257075746+193851748+138040750+91681752+56530754+31633756+15913758+6887760+2  
508262+685264+133266+11268  
DENOMINATOR:  
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)

D=11; J=3  
NUMERATOR:  
+974+6176+26278+770710+1863712+3709714+6596716+10459718+15366720+20812722+26538724+31595726+35635728+37787730+38081732+36148734+32587736  
+27549738+22018740+16373742+11408744+7254746+4228748+2143750+946752+329754+87256+12758+1760  
DENOMINATOR:  
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

D=11; J=4  
NUMERATOR:  
+328+6210+19712+27214+50716+67718+84720+89722+97724+84226+74728+54730+36732+20734+12736+2738+1740  
DENOMINATOR:  
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

D=11; J=5  
NUMERATOR:  
+1730  
DENOMINATOR:  
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=11\*\*\*\*\*:

NUMERATOR:  
+170+174-176+1078+11710+58712+85714+222716+336718+660720+951222+1589724+2154226+3188728+4080730+5510732+6633734+8310736+9443738+11059740  
+11894742+13094744+13319746+13852248+13319750+13094752+11894754+11059756+9443758+8310760+6633762+5510764+4080766+3188768+2154770+1589772  
+951774+660776+336778+222780+85782+58784+11786+10288-1790+1792+1796

DENOMINATOR:  
 $(1-Z_4)(1-Z_6)(1-Z_8)(1-Z_{10})(1-Z_{12})(1-Z_{14})(1-Z_{16})(1-Z_{18})(1-Z_{20})$

D=12; J=0  
 NUMERATOR:  
 $+1Z_0+1Z_1+10Z_2+26Z_3+55Z_4+106Z_5+181Z_6+276Z_7+393Z_8+520Z_9+647Z_{10}+764Z_{11}+865Z_{12}+933Z_{13}+972Z_{14}+976Z_{15}+943Z_{16}+876Z_{17}+784Z_{18}+667Z_{19}+541Z_{20}+413Z_{21}+296Z_{22}+194Z_{23}+117Z_{24}+63Z_{25}+29Z_{26}+11Z_{27}+4Z_{28}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

D=12; J=1  
 NUMERATOR:  
 $+5Z_1+20Z_2+66Z_3+168Z_4+372Z_5+717Z_6+1279Z_7+2106Z_8+3268Z_9+4792Z_{10}+6716Z_{11}+8991Z_{12}+11595Z_{13}+14397Z_{14}+17288Z_{15}+20080Z_{16}+22533Z_{17}+24729Z_{18}+2626Z_{19}+27094Z_{20}+27179Z_{21}+25482Z_{22}+25099Z_{23}+23082Z_{24}+20624Z_{25}+17854Z_{26}+14979Z_{27}+12135Z_{28}+9498Z_{29}+7136Z_{30}+5149Z_{31}+3540Z_{32}+2314Z_{33}+1417Z_{34}+814Z_{35}+424Z_{36}+2017Z_{37}+81Z_{38}+27Z_{39}+5Z_{40}+1Z_{41}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

D=12; J=2  
 NUMERATOR:  
 $+1Z_1+7Z_2+17Z_3+37Z_4+68Z_5+112Z_6+161Z_7+221Z_8+278Z_9+332Z_{10}+369Z_{11}+396Z_{12}+394Z_{13}+382Z_{14}+340Z_{15}+293Z_{16}+234Z_{17}+179Z_{18}+120Z_{19}+80Z_{20}+43Z_{21}+22Z_{22}+7Z_{23}+3Z_{24}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

D=12; J=3  
 NUMERATOR:  
 $+1Z_2+2Z_3+9Z_4+15Z_5+26Z_6+35Z_7+52Z_8+60Z_9+74Z_{10}+77Z_{11}+80Z_{12}+72Z_{13}+69Z_{14}+52Z_{15}+42Z_{16}+29Z_{17}+19Z_{18}+8Z_{19}+6Z_{20}+1Z_{21}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

D=12; J=4  
 NUMERATOR:  
 $+1Z_5+2Z_7+2Z_8+3Z_9+2Z_{10}+2Z_{11}+1Z_{12}+2Z_{13}+1Z_{14}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

D=12; J=5  
 NUMERATOR:  
 $+1Z_{15}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

\*\*\*\*\*THE POINCARÉ SERIES FOR D=12\*\*\*\*\*  
 NUMERATOR:  
 $+1Z_0+1Z_1+1Z_2+1Z_3+1Z_4+1Z_5+1Z_6+1Z_7+1Z_8+1Z_9+1Z_{10}+1Z_{11}+1Z_{12}+1Z_{13}+1Z_{14}+1Z_{15}+1Z_{16}+1Z_{17}+1Z_{18}+1Z_{19}+1Z_{20}+1Z_{21}+1Z_{22}+1Z_{23}+1Z_{24}+1Z_{25}+1Z_{26}+1Z_{27}+1Z_{28}$   
 DENOMINATOR:  
 $(1-Z_1)(1-Z_2)(1-Z_3)(1-Z_4)(1-Z_5)(1-Z_6)(1-Z_7)(1-Z_8)(1-Z_9)(1-Z_{10})(1-Z_{11})(1-Z_{12})(1-Z_{13})(1-Z_{14})(1-Z_{15})(1-Z_{16})(1-Z_{17})(1-Z_{18})(1-Z_{19})(1-Z_{20})(1-Z_{21})(1-Z_{22})(1-Z_{23})(1-Z_{24})(1-Z_{25})(1-Z_{26})(1-Z_{27})(1-Z_{28})$

D=13; J=0  
 NUMERATOR:  
 $+1Z_0+23Z_1+376Z_2+2870Z_3+15123Z_4+60700Z_5+201500Z_6+574964Z_7+1458029Z_8+3350844Z_9+7098549Z_{10}+14019844Z_{11}+26071711Z_{12}+45976366Z_{13}+7736784Z_{14}+124821533Z_{15}+193880853Z_{16}+290870261Z_{17}+422704305Z_{18}+596397419Z_{19}+818633011Z_{20}+1094992432Z_{21}+1429391191Z_{22}+1823198363Z_{23}+2274785102Z_{24}+2778812117Z_{25}+3326200519Z_{26}+3903870722Z_{27}+4495366115Z_{28}+5081193883Z_{29}+5640128954Z_{30}+6150055700Z_{31}+6589717995Z_{32}+6939720382Z_{33}+7184275874Z_{34}+7311905549Z_{35}+7316702593Z_{36}+7198328171Z_{37}+6962478228Z_{38}+6620024603Z_{39}+6186658269Z_{40}+5681349032Z_{41}+5125446905Z_{42}+4540871714Z_{43}+394$

9131842Z88+3369733043790+2319539516292+2311771083794+1855943697296+1457563177298+11186310122100+8378993442102+6116977292104+434483358210  
6+2996901812108+2002598197113+1292942392112+803799362114+479332202116+272816922118+147364882120+74966772122+35604032124+15590052126+6203  
502128+2192972130+670907132+169152134+33232136+4402138+332140

DENOMINATOR:

(1-24)(1-26)(1-28)(1-710)(1-212)(1-714)(1-216)(1-218)(1-220)(1-222)(1-224)(1-22)

D=13; J=1

NUMERATOR:

+4222+64124+484026+2404078+91109710+282918212+756462214+1794004216+3861266218+7659540220+14180466222+24722428724+40897719226+64559801228  
+97718840730+142356964232+200254644234+272702924236+360324529236+462769288240+578640027242+705295968744+836996271246+974894735248+110744  
9465250+123060907252+1336384239254+1425186900256+1486402715258+1518598931260+1520028679262+1490553557264+1431884069266+1347206811268+12  
41135432720+1119113130272+987183163274+851329821776+717234070278+589748950280+472786093282+369031893284+280044645286+206220207288+147052  
452290+101270775292+67152619294+42708038796+25935191298+149495912100+81232622102+41218672104+19310202106+8217602108+3112662110+101665211  
2+274372114+56672115+7932118+542120+12122

DENOMINATOR:

(1-22)(1-24)(1-26)(1-28)(1-710)(1-212)(1-214)(1-216)(1-218)(1-220)(1-22)(1-224)

D=13; J=2

NUMERATOR:

+1922+32324+245026+1108028+37352210+100760212+232740214+474244216+877302218+1496115220+2387579222+3595701224+5153833226+7064297228+93056  
79230+11912566232+14493758234+17216780236+19838851238+22195506240+24139617242+25530689244+26275954246+26313847248+25648539250+2432196825  
2+22436202254+20115136256+17517472258+14795406260+12105258252+9571389264+7298860266+5347939268+3752016270+2504759272+1581316774+93392527  
6+510391278+253052280+111544282+41978284+12923286+2917286+431790+26292

DENOMINATOR:

(1-22)(1-24)(1-24)(1-22)(1-28)(1-210)(1-24)(1-214)(1-216)(1-22)(1-220)(1-222)

D=13; J=3

NUMERATOR:

+122+5324+49976+278828+10959210+34202212+89031214+202328216+410514218+760221220+1301267222+2085019224+3150499226+4524096228+6200838230+8  
150008232+102+7221234+12542590236+14748164238+16771885240+18459458242+19686236244+20346716246+20393381248+19815348250+18667267252+170353  
13254+15053430756+12862036258+10616110260+8445590262+6464802264+4744696765+3328278268+2218624270+1397724772+824281274+450860776+22512977  
8+100988780+39527282+13098284+3436286+661288+74290+3292

DENOMINATOR:

(1-22)(1-24)(1-26)(1-24)(1-26)(1-28)(1-210)(1-212)(1-22)(1-216)(1-218)(1-220)

D=13; J=4

NUMERATOR:

+124+1326+11328+500210+1726212+4698214+11102216+22854218+42814220+73244222+116948224+174808226+247443228+332087230+425609232+521008234+6  
12010736+689659238+747365240+779577242+782912244+756152246+703365248+628585250+540083252+444497254+350481256+263309258+188354260+1272252  
62+80987264+47864266+26192258+12890270+5680272+2119274+665276+150279+25280+1282

DENOMINATOR:

(1-22)(1-24)(1-26)(1-28)(1-24)(1-26)(1-28)(1-22)(1-212)(1-214)(1-216)(1-218)

D=13; J=5

NUMERATOR:

+1210+2212+13214+28216+58218+101220+174222+245224+347226+438228+532230+591232+646234+638236+620238+552240+473242+369244+288246+191248+12  
4250+71252+38254+13256+7258+1260

DENOMINATOR:

(1-22)(1-24)(1-22)(1-28)(1-210)(1-24)(1-22)(1-28)(1-210)(1-24)(1-214)(1-216)

D=13; J=6

NUMERATOR:

+1242

DENOMINATOR:

(1-22)(1-24)(1-26)(1-28)(1-210)(1-212)(1-24)(1-26)(1-28)(1-210)(1-212)(1-214)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=13\*\*\*\*\*:

NUMERATOR:

+120-122+224-226+2128+11210+145212+195214+721216+1192218+2923720+4818722+9726224+1539226+27097228+41017730+65481232+94216234+139755236+  
191657238+267221240+350745242+463798244+583878246+737277248+892483250+1080368252+1260687254+1467752256+1653141258+1855934260+2020122762+

2190088264+2306118266+2417065768+2462771270+2497692272+2452771274+2417065776+2306118278+2190088280+2020122782+1855934784+1653141786+1467  
752288+1260687290+1080368292+892483294+737277296+583878298+4637982100+3507452102+2572212104+1916572106+1397552108+942162110+654817112+41  
0172114+270972116+153922118+97262120+48182122+29232124+11922126+7212128+1952130+1452132+112134+212136+22138+22140+12142+12144  
DENOMINATOR:  
(1-72)(1-28)(1-210)(1-212)(1-212)(1-214)(1-216)(1-218)(1-220)(1-222)(1-224)

D=14; J=0  
NUMERATOR:  
+170+321+2822+11323+39024+107825+271726+604727+1257028+2415029+43995210+75811211+125325212+198563213+364184214+450521215+648585216+90783  
8217+1240034718+1653185219+2156821220+2754202221+3448978222+4236090223+5110360724+6055854225+7057225226+8087789727+9123203228+1012901072  
9+11076417230+11928639231+12658848232+13235243733+13639593734+13851565235+13867020736+13680951237+13305035238+12750170739+12042142740+11  
203717241+10269685242+9268538743+8236946244+7202463245+6196750246+5240713247+4355569248+355293249+2848037250+2235868251+1719873252+1293  
490253+950952254+681331255+475649756+322161257+211652258+134005259+81788260+47603261+26467262+13806263+6783264+3034265+1258266+444267+14  
1268+33269+7770  
DENOMINATOR:  
(1-22)(1-23)(1-24)(1-25)(1-26)(1-21)(1-28)(1-29)(1-210)(1-211)(1-212)(1-213)(1-22)

D=14; J=1  
NUMERATOR:  
+721+2822+9423+24324+52925+102026+178227+286528+430429+6098210+8223211+10621212+13220213+15916214+18607215+21163216+23453217+25339218+26  
703219+27438220+27497221+26869222+25608223+23798224+21567225+19045226+16370227+13663228+11044729+8605230+6431231+4578232+3081233+1937234  
+1128235+595236+276237+112238+37239+7240+1241  
DENOMINATOR:  
(1-21)(1-21)(1-21)(1-22)(1-25)(1-21)(1-27)(1-24)(1-23)(1-25)(1-211)(1-22)(1-213)

D=14; J=2  
NUMERATOR:  
+321+2022+10223+31274+84525+190126+392627+731028+1283629+21045210+32964211+49133212+70572213+97463214+130525215+169144216+213385217+2616  
61218+313191219+36539220+416937221+464601222+506927223+540826224+565378225+578240226+579723227+56868228+546755229+514225230+473792231+  
426634232+376334233+323460234+271919235+222605236+177718237+137675238+103694239+75410240+53054741+35765742+23168243+14186744+8261745+444  
7246+2230247+989748+397249+122250+33251+4252  
DENOMINATOR:  
(1-21)(1-22)(1-22)(1-23)(1-24)(1-21)(1-26)(1-27)(1-28)(1-29)(1-22)(1-211)(1-212)

D=14; J=3  
NUMERATOR:  
+322+1223+3774+9625+19426+35627+59328+91729+1314210+1903211+2332212+2907213+3466214+3997215+4423216+4764217+4931218+4959219+4809220+4534  
721+4102222+3612223+3042224+2480225+19247226+1438227+1001228+670229+407730+230231+111732+52233+15234+5735  
DENOMINATOR:  
(1-21)(1-21)(1-23)(1-21)(1-23)(1-21)(1-25)(1-23)(1-27)(1-22)(1-29)(1-25)(1-211)

D=14; J=4  
NUMERATOR:  
+324+625+2026+3227+5928+9829+161210+204211+292212+340213+432214+460215+535216+529217+564218+514219+507220+425221+387222+293223+246224+16  
5225+125226+70227+48728+21229+12230+2231+1232  
DENOMINATOR:  
(1-21)(1-22)(1-21)(1-24)(1-22)(1-21)(1-24)(1-25)(1-22)(1-27)(1-28)(1-23)(1-210)

D=14; J=5  
NUMERATOR:  
+128+229+2210+2211+5212+5213+7214+7215+7216+5217+6218+4219+5220+3221+2222+1224  
DENOMINATOR:  
(1-21)(1-21)(1-23)(1-22)(1-25)(1-21)(1-23)(1-22)(1-25)(1-23)(1-27)(1-24)(1-29)

D=14; J=6  
NUMERATOR:  
+1221  
DENOMINATOR:

(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=14\*\*\*\*\*:

NUMERATOR:

+170+174-125+576+377+1828+2129+56710+7211+155712+209713+375714+523715+836716+1131717+1695718+2234719+3132720+4029721+5371722+6691723+8566724+10348725+12736726+14971727+1779728+20306729+2340730+2597331+2902332+31385733+34068734+35858735+37893736+38831737+39932738+39890739+39932740+38831741+37893742+35858743+34068744+31385745+29023746+25973747+23407748+20306749+17789750+14971751+12736752+10348753+8566754+6691755+5371756+4029757+3132758+2234759+1695760+1131761+836762+523763+375764+209765+155766+72767+56768+212769+17770+3771+5272-1773+1774+1778

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)

D=15; J=0

NUMERATOR:

+170+3822+78124+684976+3920378+153746710+548461212+1541113214+3776418716+8263874718+15485749720+30410166722+52520437224+85693905726+133120549228+198046537230+283613598732+392520604234+526850269736+687748361738+875197714740+1087755068742+1322493470744+1574771430746+1838469934746+2105978316750+2368728082752+2617407400754+2842723878756+3035720855758+3168583260760+3294853794762+3350094491764+3351871455766+3300179988768+3197160663270+304725691972+2856673194774+2633273183776+2385833516778+2123757504760+1856262169782+1592085664784+1338814362786+1102767476788+858601135790+699435601792+536745422794+400674476796+290114572798+2030816232100+1369638287102+883803137104+543503262106+316024547108+172086507110+86783972112+39905347114+16433697116+5899347118+1789222126+433347122+78477124+9607126+547128

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z14)(1-Z15)(1-Z16)(1-Z17)(1-Z18)(1-Z19)(1-Z20)(1-Z21)(1-Z22)(1-Z23)(1-Z24)(1-Z25)(1-Z26)(1-Z27)(1-Z28)(1-Z29)(1-Z30)(1-Z31)(1-Z32)(1-Z33)(1-Z34)(1-Z35)(1-Z36)(1-Z37)(1-Z38)(1-Z39)(1-Z40)(1-Z41)(1-Z42)(1-Z43)(1-Z44)(1-Z45)(1-Z46)(1-Z47)(1-Z48)(1-Z49)(1-Z50)(1-Z51)(1-Z52)(1-Z53)(1-Z54)(1-Z55)(1-Z56)(1-Z57)(1-Z58)(1-Z59)(1-Z60)(1-Z61)(1-Z62)(1-Z63)(1-Z64)(1-Z65)(1-Z66)(1-Z67)(1-Z68)(1-Z69)(1-Z70)(1-Z71)(1-Z72)(1-Z73)(1-Z74)(1-Z75)(1-Z76)(1-Z77)(1-Z78)(1-Z79)(1-Z80)(1-Z81)(1-Z82)(1-Z83)(1-Z84)(1-Z85)(1-Z86)(1-Z87)(1-Z88)(1-Z89)(1-Z90)(1-Z91)(1-Z92)(1-Z93)(1-Z94)(1-Z95)(1-Z96)(1-Z97)(1-Z98)(1-Z99)(1-Z100)

D=15; J=1

NUMERATOR:

+7722+166574+1665776+10548678+505042210+1928873212+6247203714+17744574716+45292581718+105731137220+228818638722+463891390724+888416610726+1618066525728+2818103069730+4714842308732+7606556317734+11871573921236+17972843588238+25455901838740+37940223493742+53100591875744+72640319855746+97253260012748+127579383219250+164151439424752+207341021126254+257301673609256+313920660166258+376775891173760+445111804407262+517827740590764+593493232023766+670378977844768+746517181198770+819774233145772+887948109083774+94886797703776+1000507687035778+1041036864349780+1069172159326282+1083751285539784+1084295176842786+1670782617643288+1043706829951790+1004042139671792+953194507389794+892918474434796+825227267729798+7522622530127100+676288956612102+5993877454087104+523539944407106+4505658930457108+3818500235612110+3185416435037112+2614241369337114+2109433083962116+1672354614382118+1301649006807120+993757061592122+743448222607124+544392231377126+389669717107128+272244810007130+185333153407132+122692775557134+78804793617136+48976594312138+29359232672140+16912050057142+9319313647144+4886132937146+2421312017143+1124878927150+464873007152+19139327154+67982332156+21229587158+5636157160+1212067162+194807164+20397166+997168+12170

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z26)(1-Z28)(1-Z30)(1-Z32)(1-Z34)(1-Z36)(1-Z38)(1-Z40)(1-Z42)(1-Z44)(1-Z46)(1-Z48)(1-Z50)(1-Z52)(1-Z54)(1-Z56)(1-Z58)(1-Z60)(1-Z62)(1-Z64)(1-Z66)(1-Z68)(1-Z70)(1-Z72)(1-Z74)(1-Z76)(1-Z78)(1-Z80)(1-Z82)(1-Z84)(1-Z86)(1-Z88)(1-Z90)(1-Z92)(1-Z94)(1-Z96)(1-Z98)(1-Z100)

D=15; J=2

NUMERATOR:

+4122+106074+1159076+7608478+363753210+1379223712+4395902714+12205125216+3031674718+68631733220+143680206722+281208604724+519112010726+910117301728+1524180259730+2449590437732+3792955851234+5676733192736+8235059285236+11606377290740+15924599364742+21307234966744+2784348461246+3558057258248+44512542767750+54569328702752+65611156426754+77425165867256+89729542913258+102180707614260+114388696219762+125934307635764+136393172282766+145358481676768+15246754096770+157422870783772+160613467539774+160126870092776+157758647431778+153010743226280+146086584977782+137275641621784+1269364452252786+115471972939288+103306826824790+90860882384792+78528325677794+66656932734796+55535179734798+453818325947100+363438232837102+284968519627104+218527710537106+163683194972108+119579329227110+85058395937112+58793874987114+39399056122116+25527256507118+19939542197120+9555419767122+5474007297124+2980168337126+1531226947128+736273357130+327593937132+133159347134+48479577136+15430737138+4138397140+889617142+139327144+13797146+567148

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z26)(1-Z28)(1-Z30)(1-Z32)(1-Z34)(1-Z36)(1-Z38)(1-Z40)(1-Z42)(1-Z44)(1-Z46)(1-Z48)(1-Z50)(1-Z52)(1-Z54)(1-Z56)(1-Z58)(1-Z60)(1-Z62)(1-Z64)(1-Z66)(1-Z68)(1-Z70)(1-Z72)(1-Z74)(1-Z76)(1-Z78)(1-Z80)(1-Z82)(1-Z84)(1-Z86)(1-Z88)(1-Z90)(1-Z92)(1-Z94)(1-Z96)(1-Z98)(1-Z100)

D=15; J=3

NUMERATOR:

+522+22074+241676+1483678+62287710+203960712+552988714+1300192716+2721380718+5184445720+9120288722+15002712724+23277924226+34327595728+48369794730+65429688732+85243577734+107277737736+130677146738+154365863240+177055486742+197423306744+214149522746+226130832748+232511504750+232857429752+227123077754+215738312756+199479169258+179463527260+156956218762+133321168264+109824373766+87598645268+67499441770+50121282772+3573298774+24365053776+15794543778+9670505280+5539535782+2936941784+1418300786+512433788+229516290+71982792+17605794+3053296+294798+97100

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)(1-Z20)(1-Z22)(1-Z8)

D=15; J=4

NUMERATOR:

+1074+18726+166128+9135710+38048212+127672214+365451716+917298718+2071378220+4272481722+8165350724+14597354726+24626480728+39464288730+6  
 9408695232+88707199734+125445704736+171322365235+226544195740+290597061742+362234328744+439349796746+519137053748+598120187250+672499705  
 252+738309854254+791877981256+836004661258+850403981760+851769818262+834055611264+798325132766+746802175768+682507592770+609127929772+53  
 0548308274+430666214776+372967180278+300412503280+235169296287+178647944784+131424903286+93421234788+63970034790+42054684792+26421779294  
 +15784859296+8904938798+47084297100+23082972102+10372442104+4195812106+1498832108+456612110+114222112+21442114+2742116+152118

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)(1-Z18)(1-Z20)(1-Z22)

D=15; J=5

NUMERATOR:

+126+1328+113710+500212+1726214+4698216+11102718+22854220+42814222+73244224+116948226+174808728+247443730+332087232+425609734+521003736+  
 612010238+689569240+747865242+779577244+782912746+756152248+703365250+628585252+540083254+444497256+350481758+263309760+188354762+127225  
 264+80987266+47664268+26192270+12990272+5680774+2119776+665273+150280+25282+1284

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z4)

D=15; J=6

NUMERATOR:

+1214+2716+13718+26720+58722+161724+174226+245728+347230+438232+532734+591236+646238+638240+620742+552744+473746+369748+288750+191752+12  
 4254+71756+38758+13260+7762+1264

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z6)

D=15; J=7

NUMERATOR:

+1256

DENOMINATOR:

(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=15\*\*\*\*\*:

NUMERATOR:

+120-122+374-276+3478+44210+334712+693714+2431716+5241218+13435220+27349222+58848224+111154726+212550228+374115230+654965732+1082426734+  
 1767075236+2762925138+4256995740+6339393742+9297295244+13259049745+18626874748+25553002750+34551679752+45761015254+59777469256+766597582  
 58+77029205260+120774843262+148464903764+179718255265+214966329768+253469532776+295457984772+339781570774+386448587776+433908719278+4819  
 86117280+528813856782+574135025284+615915260286+653969266288+686288827790+712929175292+732108080794+744268809796+748015934798+7442688097  
 100+7321080807102+7129291757104+6862888272106+6539692662108+6159152607110+5741350252112+5288138567114+4819861177116+4339087192118+386448  
 5877120+3397915707122+2954578847124+2534695327126+2149663297128+1797182552130+1484649037132+1207748437134+970292052136+766597587138+5977  
 74692140+457610152142+345516797144+255530027146+186268747148+132590497150+92972952152+63393937154+42569957156+27629257158+17670757160+10  
 824267164+6549657164+3741152166+2125502163+1111547170+588482172+273497174+134352176+52412178+24317180+6937182+3342184+447186+342188-2219  
 0+37192-12194+12196

DENOMINATOR:

(1-Z2)(1-Z8)(1-Z10)(1-Z12)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z26)(1-Z28)

D=16; J=0

NUMERATOR:

+170+421+3622+15373+52524+150625+380426+860477+1789078+3452979+62623210+107566711+176183212+276513213+417905714+610234715+863775716+1188  
 295217+1592457718+2082749219+2663197220+3333959221+4090910222+4925705223+5824926724+6770242225+7739567226+8706680727+9643362728+10519949  
 229+11307229730+11977441731+12506691732+12874964233+13068579234+13030029235+12908993236+12561771237+12051864738+11397891239+10623842740+  
 9756819241+8826081242+7851059243+6890686244+5940938245+5034920246+4191230247+3423931248+2742076749+2150389250+1648887251+1234307252+9002  
 68253+638390254+438915255+291773256+186791257+114700758+67195759+37323760+19482761+9484762+4230763+1705264+602265+182766+41267+8268

DENOMINATOR:

(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z1)(1-Z9)(1-Z5)(1-Z11)(1-Z3)(1-Z13)(1-Z7)(1-Z15)(1-Z2)

D=16; J=1  
 NUMERATOR:  
 +1121+7922+40223+151724+477325+1300176+317927+7095828+14718829+236483210+528281211+928867212+1566654213+2545270714+3999324215+609545321  
 6+9035971217+13056351218+18424614719+25431928720+34387351221+45600410222+59369612723+75959365224+95584689225+118383340226+144403348227+1  
 73575948228+20570886729+240468763230+277386338731+315849844232+355128926233+394381597234+432692894735+469097175236+502627550737+5323430  
 91738+557385451239+577002474240+590597620241+597745644242+598226247243+592020018244+579325932745+560535078246+536226434247+567126269248+  
 474086578249+438832199250+399935033251+360758159252+321430462253+282801486254+2456242127255+210522607256+177991787257+148380521758+121903  
 998759+98644627260+78573486261+61580874762+47404566263+35843599264+25584988265+19317953266+13734402767+9538097268+6458894769+4255029270+  
 2720475271+168267772+1003433273+574154274+313678275+162438276+79121277+35809278+14882779+5541280+1810281+491282+104283+13284+1285  
 DENOMINATOR:  
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)(1-210)(1-211)(1-212)(1-213)(1-22)(1-215)

D=16; J=2  
 NUMERATOR:  
 +421+3522+14023+42524+106925+232826+455477+816828+1359929+21271210+31506211+44480212+60195213+78438214+98764215+120563216+143018217+1651  
 79218+186011719+204451220+219464221+230189222+235960223+235474224+231508725+221576226+207211727+139291228+168774229+146756730+124284731+  
 102320232+11689733+63071234+46915235+33472236+22793237+14714238+8931239+5052740+2621741+1225242+503743+173744+43245+8246  
 DENOMINATOR:  
 (1-21)(1-21)(1-21)(1-21)(1-22)(1-25)(1-21)(1-27)(1-24)(1-23)(1-25)(1-211)(1-22)(1-213)(1-27)

D=16; J=3  
 NUMERATOR:  
 +922+6223+28324+95425+270326+557127+1442928+2899179+54280210+95604211+160000212+255586213+392668214+581551215+833770216+1160136217+15708  
 07218+2072980219+2671551220+3366270221+4152587222+5019504223+5951135224+6924585225+7913422226+8885860227+9808874228+10647655229+11370122  
 230+11945814731+12351438732+12568707233+12588805234+12410181735+12041460736+11497621737+10802545738+9984325739+9075639240+8110100241+712  
 2278242+143090243+5201421244+4319875245+3516539246+2802773247+2135046248+1663606749+1235321250+892730251+626603752+425881753+279513254+  
 176283255+106418256+61051257+33052258+16680259+7770260+3244261+1198262+367263+88264+12765+1766  
 DENOMINATOR:  
 (1-21)(1-22)(1-23)(1-22)(1-23)(1-24)(1-21)(1-26)(1-27)(1-28)(1-29)(1-22)(1-211)(1-212)(1-213)

D=16; J=4  
 NUMERATOR:  
 +323+1224+3725+9626+19427+35628+59329+917210+1314211+1803212+2332713+2967214+3466215+3997216+4423217+4764218+4931219+4959220+4809221+453  
 4222+4102223+3612224+3042225+2480226+1924227+1438228+1001229+670230+467231+230232+111233+52734+15235+5736  
 DENOMINATOR:  
 (1-21)(1-21)(1-23)(1-21)(1-21)(1-23)(1-21)(1-25)(1-23)(1-27)(1-22)(1-29)(1-25)(1-211)(1-23)

D=16; J=5  
 NUMERATOR:  
 +175+226+1327+2828+6229+115210+209211+329712+520713+738214+1035215+1363216+1757217+2150218+2594719+2981220+3368721+3661222+3907223+40092  
 24+4055225+3938726+3763227+3460228+3128729+2707230+2313231+1877232+1498233+1132234+838235+576236+395737+241238+144239+76240+39241+13242+  
 7243+1244  
 DENOMINATOR:  
 (1-21)(1-22)(1-21)(1-24)(1-25)(1-22)(1-21)(1-24)(1-25)(1-22)(1-27)(1-28)(1-23)(1-210)(1-211)

D=16; J=6  
 NUMERATOR:  
 +1711+2712+2713+2714+5215+5216+7217+7218+7219+5220+6221+4222+5223+3724+2725+1727  
 DENOMINATOR:  
 (1-21)(1-21)(1-23)(1-22)(1-25)(1-23)(1-21)(1-23)(1-22)(1-25)(1-23)(1-27)(1-24)(1-29)(1-25)

D=16; J=7  
 NUMERATOR:  
 +1228  
 DENOMINATOR:  
 (1-21)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-22)(1-23)(1-24)(1-25)(1-26)(1-27)(1-28)(1-29)

\*\*\*\*\*THE POINCARÉ SERIES FOR D=16\*\*\*\*\*  
 NUMERATOR:  
 +170+173+174+275+826+1127+3028+5929+112210+204211+389212+655213+1146214+1896215+3067216+4823217+7477218+11167219+16506220+23764221+33593

722+46576723+63588Z24+85049Z25+112273Z26+145668Z27+186298Z28+234773Z29+292016Z30+357883Z31+433493Z32+518058Z33+611720Z34+713572Z35+82299  
7236+93774Z37+1057289Z38+1178403Z39+1299472Z40+1417560Z41+1530508Z42+1634576Z43+1728646Z44+1808808Z45+1873856Z46+1921520Z47+1951002Z48+  
1960390Z49+1951002Z50+1921520Z51+1873856Z52+1808808Z53+1728646Z54+1634576Z55+1530508Z56+1417560Z57+1299472Z58+1178403Z59+1057289Z60+9377  
42Z61+82299Z62+713572Z63+611720Z64+518058Z65+433493Z66+357883Z67+292016Z68+234773Z69+186298Z70+145668Z71+112273Z72+85049Z73+63588Z74+46  
576Z75+33593Z76+23764Z77+16505Z78+11167Z79+7477Z80+4823Z81+3067Z82+1896Z83+1146Z84+655Z85+389Z86+204Z87+112Z88+59Z89+30Z90+11Z91+8Z92+2Z  
93+1Z94+1Z95+1Z98

DENJMINATOR:

(1-72)(1-74)(1-75)(1-76)(1-76)(1-77)(1-77)(1-78)(1-79)(1-710)(1-711)(1-712)(1-713)(1-715)





